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Global Asymptotics in Some Quasimonotone Reaction-Diffusion Systems with Delays

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Both dichotomy and trichotomy on the global asymptotics of some quasimonotone reaction-diffusion systems with delays are established in terms of the principal eigenvalue of linear weakly coupled elliptic systems. Applications to a class of

1. INTRODUCTION

In [16], Martin and Smith proved that a quasimonotone reaction-diffusion system with time delays generates a monotone semiflow on a suitable phase space. They also obtained additional conditions for this semiflow to be eventually strongly monotone. In order to understand the above and what follows, we make precise the following concepts.

DEFINITIONS. Let X be an ordered Banach space with a positive cone P satisfying $\operatorname{int}(P) \neq \emptyset$, let $S: P \to P$ be a continuous map, and let $x, y \in X$. We say that

(i) $x \geqslant y$ if $x - y \in P$;

(ii) $x > y$ if $x - y \in P \backslash \{0\}$;

(iii) $x \gg y$ if $x - y \in \operatorname{int}(P)$;

(iv) S is *monotone* if $x \geqslant y \Rightarrow S(x) \geqslant S(y)$;

(v) S is *strongly monotone* if $x > y \Rightarrow S(x) \gg S(y)$;

(vi) S is *subhomogeneous* if $x \geqslant 0$ and $\alpha \in (0, 1) \Rightarrow S(\alpha x) \geqslant \alpha S(x)$;

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(vii) S is *strictly subhomogeneous* if $x \gg 0$ and $\alpha \in (0, 1) \Rightarrow S(\alpha x) > \alpha S(x)$;

(viii) S is *strongly subhomogeneous* if $x > 0$ and $\alpha \in (0, 1) \Rightarrow S(\alpha x) \gg \alpha S(x)$.

The terminology "subhomogeneous" is used in [12], [13] and [23]. In [10], [13] and [22], this concept is also referred to as "sublinear."

Stability and convergence results were obtained by several authors utilizing various combinations of the above definitions. In [10], [12] and [22], strong monotonicity was combined with subhomogeneity. In [12] also monotonicity was combined with strong subhomogeneity. In [25] strong monotonicity was combined with strict subhomogeneity in order to obtain uniqueness of positive fixed points. In addition, global stability was also studied. As noted in [25], both Krasnoselskii's strong concavity and Smith's concept of concavity in [18] imply strict subhomogeneity. Moreover, Krause and Nussbaum [13] proved a limit set trichotomy for part metric contractive maps on a solid and normal cone in Banach space, and made an interesting observation that a monotone map with strong subhomogeneity is contractive for the part metric on the interior of the cone. The concept of part metric was also utilized for convergence in [23]. In [14], Martin also proved a result on global attractivity for quasimonotone reaction-diffusion systems subject to Dirichlet boundary conditions under the assumptions of both strict subhomogeneity and a one-sided monotone Lipschitz condition on the reaction term (see (C3) and (C4) in [14, Theorem 7]). More recently, the global stability of cooperative systems of functional differential equations with strict subhomogeneity was studied in [26].

The object of this paper is to study the global asymptotic behavior in some quasimonotone reaction-diffusion systems with delays. For such systems with strict subhomogeneity, we establish a trichotomy of the global dynamics by taking as the threshold parameter the principal eigenvalue of the eigenvalue problem associated with the linearization as the trivial solution. More precisely, the semiflow generated by our reaction-diffusion system with delays is only monotone and subhomogeneous and the semiflow generated by its corresponding reaction-diffusion system without delays is strictly subhomogeneous. Even in our Proposition 3.3, we don't need the semiflow generated by the reaction-diffusion system with delays to be subhomogeneous. Therefore we are unable to directly use any of the results on monotone maps mentioned in the previous paragraphs since none of them combines monotonicity with strict subhomogeneity.

The organization of this paper is as follows. In Section 2, by the general theory of quasimonotone reaction-diffusion systems with delays, the principal eigenvalue theory and monotone dynamical system theory, we first prove a dichotomy on the global dynamics under the assumption of instability of the trivial solution (Theorem 2.1). Then with the assumption of strict subhomogeneity and taking the principal eigenvalue as the threshold parameter, we further establish our main result on the trichotomy of the global dynamics (Theorem 2.2). In many application, Theorem 2.2 can provide a threshold result on the global attractivity of either the zero solution or a unique positive steady state if one can prove the existence of a bounded positive solution instead of proving the dissipativity of the system or the existence of a nonempty compact invariant set. Clearly, there are simple versions of Theorems 2.1 and 2.2 for quasimonotone reaction-diffusion systems (Corollaries 2.1 and 2.2). Corollary 2.2 also generalizes a threshold type result [25, Theorem 3.4] for the scalar Kolmogorov reaction-diffusion equations and a related result [14, Theorem 7] for the quasimonotone reaction-diffusion systems subject to the Dirichlet boundary condition (Remark 2.1).

Section 3 is aimed at some applications. By a comparison method and the celebrated Krein–Rutman theorem, we first prove two properties of the principal eigenvalue of weakly coupled elliptic eigenvalue problems (Propositions 3.1 and 3.2). These two results also have their own interest and are useful in the estimation of the principle eigenvalue. Then we obtain a threshold result on the global dynamics (Proposition 3.3) and a criterion for uniform persistence (Remark 3.2) for a class of delayed reaction-diffusion models of single species growth. For a reaction-diffusion system with delay modelling the spread of bacterial infections, we also prove a threshold result for the global dynamics (Proposition 3.4), which, in the case without delay, improves the main results in [3, Theorems 5.1 and 5.5] and [4, Theorems 4.2 and 5.6] in the sense that we only use one threshold parameter and the assumption of strict subhomogeneity (Remark 3.5).

2. GLOBAL ASYMPTOTICS

Suppose that Ω is a bounded domain in R^N with a smooth boundary $\partial\Omega$, and that $\tau=(\tau_1, ..., \tau_n)$ is a vector in $R_n^+=\{(u_1, ..., u_n); u_i \geqslant 0, 1\leqslant i\leqslant n\}$. Let $|\tau|=\max_{1\leqslant i\leqslant n}\tau_i$, and define $C_\tau=\prod_{i=1}^n C([-\tau_i, 0], R)$. For any $\phi=(\phi_1, ..., \phi_n)\in C_\tau$, define $\|\phi\|=\sum_{i=1}^n \|\phi_i\|_\infty$, where $\|\phi_i\|_\infty=\max_{\theta\in[-\tau_i, 0]}|\phi_i(\theta)|$. Let $C_\tau^+=\{(\phi_1, ..., \phi_n)\in C_\tau;\ \phi_i(\theta)\geqslant 0,\ 1\leqslant i\leqslant n, \theta\in[-\tau_i, 0]\}$, then C_τ^+ is a normal cone of C_τ with nonempty interior in C_τ, and hence (C_τ, C_τ^+) is an ordered Banach space. Let $X_\tau=\prod_{i=1}^n C([-\tau_i, 0],\ C(\bar{\Omega}, R))$. We will identify X_τ with $X_\tau=\prod_{i=1}^n C(\bar{\Omega}\times[-\tau_i, 0], R)$ where this does not cause confusion. For any $\phi=(\phi_1, ..., \phi_n)\in X_\tau$, define $\|\phi\|=\sum_{i=1}^n \|\phi_i\|_\infty$, where $\|\phi_i\|_\infty=\max_{(x, \theta)\in\bar{\Omega}\times[-\tau_i, 0]}|\phi_i(x, \theta)|$. Let $X_\tau^+=\{(\phi_1, ..., \phi_n)\in X_\tau;\ \phi_i(x, \theta)\geqslant 0,$

$1 \leqslant i \leqslant n,\ x \in \bar{\Omega},\ \theta \in [-\tau_i, 0]\}$, then X_τ^+ is a normal cone of X_τ with non-empty interior in X_τ, and hence (X_τ, X_τ^+) is an ordered Banach space. Let $\hat{}$ denote the inclusion $R^n \to C_\tau$ by $u \to \hat{u}$, $\hat{u}_i(\theta) \equiv u_i$, $\theta \in [-\tau_i, 0]$, $i = 1, 2, ..., n$. Given the function $u_i(x, t)$ defined for $x \in \bar{\Omega}$ and $t \in [-\tau_i, \sigma)$, $\sigma > 0$, and $0 \leqslant t < \sigma$, define $u_t \in C_\tau$ by $u_t = (u_t^1, ..., u_t^n)$, where $u_t^i(x, \theta) = u_i(x, t+\theta)$, $x \in \bar{\Omega}$, $\theta \in [-\tau_i, 0]$, $1 \leqslant i \leqslant n$.

Consider the reaction-diffusion system with delays

$$\begin{cases} \dfrac{\partial u}{\partial t} = D\Delta u + F(x, u_t(x)), & t > 0, \quad x \in \Omega \\ \dfrac{\partial u}{\partial \nu}(x, t) + \alpha(x)\, u(x, t) = 0, & t > 0, \quad x \in \partial\Omega \\ u_0 = \phi \in X_\tau, \end{cases} \tag{2.1}$$

where $F\colon \bar{\Omega} \times C_\tau \to R^n$ is continuous and Lipschitz on bounded subsets of C_τ uniformly in $x \in \bar{\Omega}$, Δ is the Laplacian operator on Ω, D is the diagonal matrix with $d_i > 0$ on the diagonal, $\alpha(x)$ is a diagonal matrix with $\alpha(\cdot) \in C^1(\bar{\Omega}, [0, \infty))$ on the diagonal and $\partial/\partial\nu$ denotes differentiation in the direction of the outward normal to $\partial\Omega$.

According to [16], for any $\phi \in X_\tau$, (2.1) admits a unique mild solution $u(t, \phi)$, defined on its maximal interval of existence $[0, \sigma_\phi)$, $\sigma_\phi > 0$, satisfying $u_0 = \phi$. Moreover, if $|\tau| < \sigma_\phi$, then for $t > |\tau|$, $u(x, t) = u(t, \phi)(x)$ is a classical solution of (2.1).

We will impose the following quasimonotone condition on F:

(QM) Whenever $\phi, \psi \in C_\tau$ satisfy $\phi \leqslant \psi$ and $\phi_i(0) = \psi_i(0)$ for some i, then $F_i(x, \phi) \leqslant F_i(x, \psi)$ for all $x \in \bar{\Omega}$.

Let $BL(C_\tau, R^n)$ be the linear normed space of all bounded linear maps from C_τ into R^n. For any $L \in C(\bar{\Omega}, BL(C_\tau, R^n))$, $L = (L_1, ..., L_n)$ admits the following standard representation

$$L_i(x)\,\phi = \sum_{j=1}^{n} \int_{-\tau_j}^{0} \phi_j(\theta)\, d_\theta \eta_{ij}(\theta, x),\ x \in \bar{\Omega},\ \phi \in C_\tau,$$

where $\eta_{ij}(\cdot, x)\colon R \to R$ satisfies, for each $x \in \bar{\Omega}$, $1 \leqslant i, j \leqslant n$,

(i) $\eta_{ij}(\theta, x) = \eta_{ij}(0, x)$ for $\theta \geqslant 0$, $\eta_{ij}(\theta, x) = 0$ for $\theta \leqslant -\tau_j$;

(ii) $\eta_{ij}(\cdot, x)$ is of bounded variation on $[-\tau_j, 0]$ and $\eta_{ij}(\cdot, x)$ is continuous from the left on $(-\tau_j, 0)$.

We will further require the following condition:

(R) For each j for which $\tau_j > 0$, there exists i such that $\eta_{ij}(\theta, x) > 0$ for all $\theta \in (-\tau_j, 0)$, $x \in \bar{\Omega}$.

Let (X, P) be an ordered Banach space. As mentioned previously, a continuous semiflow $S(t)\colon U \subset X \to X$, $t \geqslant 0$, is called monotone if for any $t \geqslant 0$ and $x, y \in U$ with $x \geqslant y$, $S(t)\, x \geqslant S(t)\, y$. For a general theory of monotone semiflows, we refer to the recent monograph [21]. We will need the following lemma which comes from [26, Theorem 2.2]. For a similar result on the discrete monotone semiflow (i.e., the iterations of a monotone map), we refer to [19, Theorem 2.1].

LEMMA 2.1. *Let P be a normal cone of Banach space X with* $\operatorname{int}(P) \neq \varnothing$. *Assume that*

(1) *$S(t)\colon P \to P$, $t \geqslant 0$, is a monotone C^0-semiflow and $S(t)\, 0 = 0$ for all $t \geqslant 0$;*

(2) *there exists $t_0 > 0$ such that $S(t)\, u \gg 0$ for any $u \gg 0$ and $t \geqslant t_0$ and for $S = S(t_0)\colon P \to P$, every bounded positive orbit in P is precompact, the Fréchet derivative $DS(0)$ exists and is compact and strongly positive, and its spectral radius $r = r(DS(0)) > 1$.*

Then either

(a) *for any $u > 0$, $\lim_{t \to \infty} \|S(t)\, u\| = +\infty$, or alternatively,*

(b) *there exists $u^* \gg 0$ with $S(t)\, u^* = u^*$ for all $t \geqslant 0$ such that for any $0 < u \leqslant u^*$, $\lim_{t \to \infty} S(t)\, u = u^*$.*

Assume that the continuous $n \times n$ matrix $M(x) = (m_{ij})$, $x \in \bar{\Omega}$, satisfies the following cooperative and irruducible conditions:

(CR) (i) $m_{ij}(x) \geqslant 0$, $i \neq j$, $x \in \bar{\Omega}$; (ii) $M(x_0)$ is irreducible for some $x_0 \in \Omega$.

Then, according to [21, Theorem 7.6.1], the eigenvalue problem

$$\begin{cases} D\Delta w(x) + M(x)\, w(x) = \lambda w(x), & x \in \Omega \\ \dfrac{\partial w(x)}{\partial \nu} + \alpha(x)\, w(x) = 0, & x \in \partial\Omega \end{cases} \tag{2.2}$$

has a unique principal eigenvalue $\lambda_0 = \lambda_0(M(\cdot))$ with a corresponding eigenfunction $w_0(x)$ satisfying $w_0 \gg 0$ in $C(\bar{\Omega}, R^n)$ (i.e., $w_0(x) \gg 0$ for all $x \in \bar{\Omega}$).

Now we are in a position to prove the following result.

THEOREM 2.1. *Let F satisfy (QM) and let $f\colon \bar{\Omega} \times R^n_+ \to R^n$ be defined by $f(x, u) = F(x, \hat{u})$, $x \in \bar{\Omega}$, $u \in R^n_+$. Assume that*

(1) *for any $\phi \in C^+_\tau$ with $\phi_i(0) = 0$, $F_i(x, \phi) \geqslant 0$, $x \in \bar{\Omega}$, and $\sigma_\phi = +\infty$ for any $\phi \in X^+_\tau$;*

(2) $F(x, \hat{0}) \equiv 0$, *and for each* $x \in \Omega$, $f_u(x, 0) = \partial f(x, 0)/\partial u$ *is irreducible and the Fréchet derivative* $L(x) = d_\phi F(x, \hat{0})$ *exists and satisfies* (R);

(3) $\lambda_0 = \lambda_0(f_u(\cdot, 0)) > 0$.

Then either

(a) *for any* $\phi \in X_\tau^+ \setminus \{\hat{0}\}$, $\lim_{t \to \infty} \|u_t(\phi)\| = +\infty$; *or alternatively*,

(b) (2.1) *admits a steady state* $\psi^* \gg 0$ *in* $C(\bar{\Omega}, R^n)$ *such that for any* $\hat{0} < \phi \leqslant \widehat{\psi^*}$, $\lim_{t \to \infty} u(t, \phi) = \psi^*$ *in* $C(\bar{\Omega}, R^n)$. *Moreover, for any* $\phi > \hat{0}$, $\liminf_{t \to \infty} u(t, \phi) \geqslant \psi^*$, *where* $\liminf_{t \to \infty} u(t, \phi) = (\liminf_{t \to \infty} u_1(t, \phi), ..., \liminf_{t \to \infty} u_n(t, \phi))$.

Proof. For any $\phi \in X_\tau^+$, by assumption (1), $u(t, \phi)$ exists globally on $[0, +\infty)$, and by [16, Proposition 1.3], $u(t, \phi)(x) \geqslant 0$, $t \geqslant 0$, $x \in \Omega$. Define $S(t)\colon X_\tau^+ \to X_\tau^+$ by

$$S(t)\, \phi = u_t(\phi), \qquad t \geqslant 0.$$

Then, by [16, Proposition 1.4], $S(t)\colon X_\tau^+ \to X_\tau^+$ is a monotone C^0-semiflow and $S(t)\, \hat{0} = \hat{0}$ for $t \geqslant 0$. Moreover, by [16, Proposition 1.2], $S(t)\colon X_\tau^+ \to X_\tau^+$ is compact for each $t > |\tau|$. We further have the following two claims.

Claim 1. For each $\phi > \hat{0}$, $S(t)\, \phi \gg \hat{0}$ in X_τ for all $t \geqslant (n+1)\, |\tau|$.

Indeed, for each $\phi \in X_\tau$, let $z(t, \phi)$ be the unique solution of the following linear reaction-diffusion system with delays,

$$\begin{cases} \dfrac{\partial z}{\partial t} = D\Delta z + d_\phi F(x, \hat{0})\, z_t(x), & t > 0, \quad x \in \Omega \\ \dfrac{\partial z(x, t)}{\partial v} + \alpha(x)\, z(x, t) = 0, & t > 0, \quad x \in \partial\Omega, \end{cases} \tag{2.3}$$

satisfying $z_0 = \phi$. Clearly, the (QM) condition of $F(x, \phi)$ implies that $G(x, \phi) \equiv d_\phi F(x, \hat{0})\, \phi$ satisfies (QM). Let $\{e_1, ..., e_n\}$ be the standard basis for R^n. It easily follows that for any $\psi \in C_\tau$, $x \in \Omega$, $d_\phi G(x, \psi) = d_\phi F(x, \hat{0})$, and

$$\begin{aligned} A(x, \psi) &\equiv d_\phi G(x, \psi)(\widehat{e_1}, ..., \widehat{e_n}) = d_\phi F(x, \hat{0})(\widehat{e_1}, ..., \widehat{e_n}) \\ &= \left.\frac{\partial F(x, \hat{u})}{\partial u}\right|_{u=0} = f_u(x, 0), \end{aligned}$$

and hence, $A(x, \psi)$ is an irreducible $n \times n$ matrix and $d_\phi G(x, \psi)$ satisfies (R). By [16, Theorem 1.5], for any $\phi, \psi \in X_\tau^+$ with $\phi < \psi$, $z_t(\phi) \ll z_t(\psi)$ for all $t \geqslant (n+1)|\tau|$. In particular, for all $t \geqslant (n+1)|\tau|$, $z_t: X_\tau^+ \to X_\tau^+$ is compact and strongly positive, i.e., for any $\phi > 0$, $z_t(\phi) \gg 0$. By a standard argument, it follows that $d_\phi u(t, \hat{0}): X_\tau \to X_\tau$ satisfies

$$d_\phi u(t, \hat{0})\, \psi = z(t, \psi), \qquad t \geqslant 0, \quad \psi \in X_\tau,$$

and hence, for any $t \geqslant |\tau|$, $\psi \in X_\tau$, $d_\phi u_t(\hat{0})\, \psi = z_t(\psi)$. Therefore for any $t \geqslant |\tau|$, $DS(t)(\hat{0}) = Du_t(\hat{0}) = z_t$. Let $t \geqslant (n+1)|\tau|$ be given. Then for any $\phi > \hat{0}$, since $DS(t)(\hat{0}) = z_t$ is strongly positive, $DS(t)(\hat{0})\, \phi \gg \hat{0}$. Since $S(t)(s\phi) = s \cdot DS(t)(\hat{0})\, \phi + o(s) = s[DS(t)(\hat{0})\, \phi + o(s)/s]$, $s > 0$, there exists $s_0 \in (0, 1]$ such that $S(t)(s_0\phi) \gg \hat{0}$. Then $S(t)\, \phi \geqslant S(t)(s_0\phi) \gg \hat{0}$.

Claim 2. Let μ_0 be the principal eigenvalue of the eigenvalue problem associated with (2.3) (see [16, Section 3]). Then for any $t \geqslant (n+1)|\tau|$, the spectral radius of $DS(t)(\hat{0}): X_\tau \to X_\tau$ satisfies $r(DS(t)(\hat{0})) = e^{\mu_0 t}$.

Indeed, by the Remark of [16, Section 3], let $v \gg 0$ in $C(\bar{\Omega}, R^n)$ be the eigenfunction corresponding to μ_0. Then $v(x, t) = e^{\mu_0 t} \cdot v(x)$ satisfies (2.3). Let $v(\cdot) = (v_1(\cdot), ..., v_n(\cdot))$, and define $\psi = (\psi_1, ..., \psi_n) \in X_\tau$ by

$$\psi_i(\theta) = e^{\mu_0 \theta} \cdot v_i(\cdot), \; -\tau_i \leqslant \theta \leqslant 0.$$

Then $\psi \gg 0$ in X_τ, and $z(t, \psi) = v(\cdot, t)$, $t \geqslant 0$. By the definition of ψ, it easily follows that

$$z_t(\psi) = e^{\mu_0 t} \cdot \psi, \; t \geqslant 0.$$

By the proof of Claim 1, $DS(t)(\hat{0}) = z_t$ for any $t \geqslant |\tau|$, and hence $DS(t)(\hat{0})$ is compact and strongly positive for any $t \geqslant (n+1)|\tau|$. Since $DS(t)(\hat{0})\, \psi = z_t(\psi) = e^{\mu_0 t}\psi$, $e^{\mu_0 t}$ is a positive eigenvalue of $DS(t)(\hat{0})$. Now the Krein–Rutman theorem (see, e.g., [1, Theorem 3.2], [10, Theorem 7.2] or [21, Theorem 2.4.1]) completes the proof of Claim 2.

Let $t_0 = (n+1)|\tau|$ and $S = S(t_0) = u_t: X_\tau \to X_\tau$. By [16, Theorem 3.1], μ_0 and λ_0 have the same signs, and hence the assumption (3) implies $r(DS(\hat{0})) = e^{\mu_0 t_0} > 1$. By Lemma 2.1, it follows that either

(a) for any $\psi > \hat{0}$, $\lim_{t \to \infty} \|S(t)\, \phi\| = \lim_{t \to \infty} \|u_t(\phi)\| = +\infty$; or alternatively,

(b) there exists $\phi^* \gg 0$ in X_τ with $u_t(\phi^*) = \phi^*$ for all $t \geqslant 0$ such that for any $\hat{0} < \phi \leqslant \phi^*$, $\lim_{t \to \infty} u_t(\phi) = \phi^*$.

In case (b), let $\psi^* = \phi^*(0) \in C(\bar{\Omega}, R^n)$. Then $u(t, \phi^*) = u_t(\phi^*)(0) = \psi^*$, $t \geqslant 0$, and hence, $\phi^* = \widehat{\psi^*}$ and ψ^* is a steady state of (2.1). Then $\lim_{t \to \infty} u(t, \phi) = \psi^*$ for any $\hat{0} < \phi \leqslant \widehat{\psi^*}$. For any $\phi > \hat{0}$, by Claim 1,

$S(t_0)\phi \gg \hat{0}$, and hence there exists $0 < \varepsilon_0 < 1$ such that $\varepsilon_0 \widehat{\psi^*} \leqslant S(t_0)\,\phi = u_{t_0}(\phi)$. Therefore

$$\liminf_{t\to\infty} u(t+t_0,\phi) = \liminf_{t\to\infty} u(t, u_{t_0}(\phi))$$
$$\geqslant \lim_{t\to\infty} u(t, \varepsilon_0 \widehat{\psi^*}) = \psi^*.$$

That is, $\liminf_{t\to\infty} u(t,\phi) \geqslant \psi^*$.

This completes the proof.

Based on Theorem 2.1, we further have the following trichotomy of the global asymptotics of system (2.1) with certain subhomogeneities.

THEOREM 2.2. *Let F satisfy (QM) and let $f\colon \bar{\Omega}\times R^n_+ \to R^n$ be defined by $f(x,u) = F(x,\hat{u})$, $(x,u) \in \bar{\Omega}\times R^n_+$. Assume that*

(1) *for any $\phi \in C^+_\tau$ with $\phi_i(0) = 0$, $F_i(x,\phi) \geqslant 0$, $x \in \bar{\Omega}$;*

(2) *for any $x \in \bar{\Omega}$, $F(x,\cdot)\colon C^+_\tau \to R^n$ is subhomogeneous, i.e., for any $\alpha \in (0,1)$, $\phi \in C^+_\tau$, $F(x,\alpha\phi) \geqslant \alpha F(x,\phi)$, and $f(x,\cdot)\colon R^n_+ \to R^n$ is strictly subhomogeneous, i.e., for any $\alpha \in (0,1)$, $u \in R^n_+$ with $u \gg 0$, $f(x,\alpha u) > \alpha f(x,u)$;*

(3) *$F(x,\hat{0}) \equiv 0$, and for each $x \in \Omega$, $f_u(x,u) = \partial f(x,u)/\partial u$ is irreducible for any $u \in R^n_+$ and the Fréchet derivative $L(x) = d_\phi F(x,\hat{0})$ exists and satisfies (R).*

Then the following trichotomy holds:

(a) *if $\lambda_0(f_u(\cdot,0)) \leqslant 0$, then $u = 0$ is globally attractive for (2.1) in X^+_τ;*

(b) *if $\lambda_0(f_u(\cdot,0)) > 0$, then either*

(b_1) *for any $\phi \in X^+_\tau \setminus \{\hat{0}\}$, $\lim_{t\to\infty} \|u_t(\phi)\| = +\infty$, or alternatively,*

(b_2) *(2.1) admit a unique positive steady state $\psi^* \gg 0$ in $C(\bar{\Omega}, R^n)$ and $u = \psi^*$ is globally attractive for (2.1) in $X^+_\tau \setminus \{\hat{0}\}$.*

Proof. Since $F(x,\hat{0}) \equiv 0$ and $F(x,\phi)$ is subhomogeneous in ϕ, for any $x \in \bar{\Omega}$, $\phi \in C^+_\tau$,

$$F(x,\phi) = \lim_{\alpha\to 0^+} \frac{\alpha F(x,\phi)}{\alpha} \leqslant \lim_{\alpha\to 0^+} \frac{F(x,\alpha\phi) - F(x,\hat{0})}{\alpha} = d_\phi F(x,\hat{0})\,\phi.$$

By [6, Theorem 4.2], the unique solution of the linear system

$$\begin{cases} \dfrac{\partial u}{\partial t} = D\Delta u + d_\phi F(x,\hat{0})\, u_t(x), & t > 0, \quad x \in \Omega \\ \dfrac{\partial u(x,t)}{\partial \nu} + \alpha(x)\, u(x,t) = 0, & t > 0, \quad x \in \partial\Omega \end{cases} \tag{2.4}$$

with $u_0=\phi\in X_\tau^+$ exists globally on $[0,+\infty)$. Then the comparison theorem [16, Theorem 2.2] implies that for any $\phi\in X_\tau^+$, the unique solution $u(t,\phi)$ of (2.1) with $u_0=\phi$ exists globally on $[0,+\infty)$. Define $S(t)\,\phi=u_t(\phi)$, $\phi\in X_\tau^+$, $t\geqslant 0$. Then $S(t)\colon X_\tau^+\to X_\tau^+$ is a monotone semiflow and $S(t)\,\hat{0}=\hat{0}$. We further claim that $S(t)\colon X_\tau^+\to X_\tau^+$ is subhomogeneous. Indeed, for any $\phi\in X_\tau^+$ and $\alpha\in(0,1)$, let $v(t)=u(t,\alpha\phi)$, $w(t)=\alpha u(t,\phi)$. By the abstract formulation of (2.1) (see [16, Section 1]), $v(t)$ and $w(t)$ satisfies

$$v(t)=T(t)\,v_0+\int_0^t T(t-s)\,\mathscr{F}(v_s)\,ds, \qquad t\geqslant 0,$$

and

$$w(t)=T(t)\,w_0+\alpha\int_0^t T(t-s)\,\mathscr{F}(u_s(\phi))\,ds, \qquad t\geqslant 0,$$

where $\mathscr{F}\colon X_\tau^+\to C(\bar{\Omega},R^n)$ is defined by $\mathscr{F}(\phi)(x)=F(x,\phi(x,\cdot))$, $x\in\bar{\Omega}$. By the subhomogeneity of $F(x,\phi)$ in ϕ, $\mathscr{F}(w_s)=\mathscr{F}(\alpha u_s(\phi))\geqslant\alpha\mathscr{F}(u_s(\phi))$. Therefore

$$w(t)\leqslant T(t)\,w_0+\int_0^t T(t-s)\,\mathscr{F}(w_s)\,ds, \qquad t\geqslant 0.$$

By the comparison theorem [15, Proposition 1 and Remarks 1.4 and 1.5], $w(t)\leqslant v(t)$, $t\geqslant 0$, i.e., $\alpha u(t,\phi)\leqslant u(t,\alpha\phi)$, $t\geqslant 0$. Therefore $\alpha S(t)\,\phi=\alpha u_t(\phi)\leqslant u_t(\alpha\phi)=S(t)(\alpha\phi)$, $t\geqslant 0$.

By our assumptions (1) and (3) and [21, Theorem 7.4.1], the parabolic reaction-diffusion system

$$\begin{cases} \dfrac{\partial u}{\partial t}=D\Delta u+f(x,u), & t>0, \quad x\in\Omega \\ \dfrac{\partial u(x,t)}{\partial \nu}+\alpha(x)\,u(x,t)=0, & t>0, \quad x\in\partial\Omega \end{cases} \tag{2.5}$$

generates a strongly monotone semiflow $\Phi(t)\colon C^+(\bar{\Omega},R^n)\to C^+(\bar{\Omega},R^n)$, defined by $\Phi(t)\,\phi=u(t,\phi)$, $t\geqslant 0$, where $u(t,\phi)$ is the unique solution of (2.5) with $u(0,\phi)=\phi$, i.e., for any $\phi>\psi$, $\Phi(t)\,\phi\gg\Phi(t)\psi$, $t>0$. In particular, since $\Phi(t)0=0$, $t\geqslant 0$, for any $\phi\gg 0$, we have $\Phi(t)\,\phi\gg 0$, $t\geqslant 0$. Moreover, we have the following claim.

Claim 1. For any $t>0$, $\Phi(t)\colon C^+(\bar{\Omega},R^n)\to C^+(\bar{\Omega},R^n)$ is strictly subhomogeneous, i.e., for any $\alpha\in(0,1)$, $\phi\gg 0$, $\Phi(t)(\alpha\phi)>\alpha\Phi(t)\,\phi$.

Indeed, for any $\phi \in C^+(\bar{\Omega}, R^n)$ any $\alpha \in (0, 1)$, let $w(t) = u(t, \alpha\phi) - \alpha u(t, \phi)$. Then $w(0) = 0$ and $w(t)$ satisfies

$$\begin{cases} \dfrac{\partial w}{\partial t} = D\Delta w + H(t, x)\, w + f(x, \alpha u(t, \phi)) - \alpha f(x, u(t, \phi)), & t > 0, \quad x \in \Omega \\ \dfrac{\partial w}{\partial \nu} + \alpha w = 0, & t > 0, \quad x \in \partial\Omega, \end{cases} \tag{2.6}$$

where

$$H(t, x) = \int_0^1 f_u(x, su(t, \alpha\phi) + (1 - s)\, \alpha u(t, \phi))\, ds.$$

Let $U(t, s)$, $t \geqslant s \geqslant 0$, be the evolution operator of the nonautonomous linear parabolic system

$$\begin{cases} \dfrac{\partial w}{\partial t} = D\Delta w + H(t, x)\, w, & t > 0, \quad x \in \Omega \\ \dfrac{\partial w}{\partial \nu} + \alpha w = 0, & t > 0, \quad x \in \partial\Omega. \end{cases} \tag{2.7}$$

By condition (3), $H(t, x)$ is a cooperative and irreducible $n \times n$ matrix for each $t \geqslant 0$ and $x \in \bar{\Omega}$. Again by [21, Theorem 7.4.1], it easily follows that $U(t, s)$, $t > s \geqslant 0$ is strongly positive, i.e., for any $\phi > 0$, $U(t, s)\, \phi \gg 0$. By the formula of variation of constants,

$$w(t) = \int_0^t U(t, s)(f(\cdot, \alpha u(s, \phi)) - \alpha f(\cdot, u(s, \phi)))\, ds, \qquad t \geqslant 0.$$

Since $u(s, \phi) \gg 0$, $s \geqslant 0$, and $f(x, u)$ is strictly subhomogeneous in u, we have

$$f(\cdot, \alpha u(s, \phi)) - \alpha f(\cdot, u(s, \phi)) > 0 \qquad \text{in} \quad C(\bar{\Omega}, R^n),$$

and hence for each $t > 0$, $w(t) \gg 0$ in $C(\bar{\Omega}, R^n)$, i.e., $\phi(t)(\alpha\phi) = u(t, \alpha\phi) \gg \alpha u(t, \phi) = \alpha\Phi(t)\, \phi$.

For the uniqueness and nonexistence of the positive steady state of (2.1), we further have the following claim.

Claim 2. (2.1) admits at most one positive steady state in X_τ^+; If $\lambda_0 = \lambda_0(f_u(\cdot, 0)) \leqslant 0$, (2.1) has no steady state in $X_\tau^+ \backslash \{\hat{0}\}$.

Indeed, it suffices to prove the corresponding conclusion for the parabolic reaction-diffusion system (2.5) associated with (2.1). Let $Z(t)$: $C(\bar{\Omega}, R^n) \to C(\bar{\Omega}, R^n)$ be the semigroup generated by the linear parabolic system

$$\begin{cases} \dfrac{\partial z}{\partial t} = D\Delta z + f_u(x, 0)z, & t > 0, \quad x \in \Omega \\ \dfrac{\partial u}{\partial \nu} + \alpha u = 0, & t > 0, \quad x \in \partial\Omega. \end{cases} \tag{2.8}$$

Then, by [17, Proposition 3.1.4] and [21, Theorem 7.4.1 and Section 7.6], for each $t > 0$, $Z(t)$: $C(\bar{\Omega}, R^n) \to C(\bar{\Omega}, R^n)$ is compact and strongly positive on $C^+(\bar{\Omega}, R^n)$, and $r(Z(t)) = e^{\lambda_0 t}$. For any given $\omega > 0$, we view (2.5) as an ω-periodic parabolic system. Then the ω-time map $P = \Phi(\omega)$: $C^+(\bar{\Omega}, R^n) \to C^+(\bar{\Omega}, R^n)$ is compact, strongly monotone and, by Claim 1, strictly subhomogeneous. Clearly, $P(0) = 0$. By [25, Lemma 1], P admits at most one positive fixed point in $C(\bar{\Omega}, R^n)$. Moreover, it easily follows that $P = \Phi(\omega)$ is Fréchet differentiable at $\phi = 0$ with $DP(0) = D\Phi(\omega)(0) = Z(\omega)$ (see, e.g., the proof of [10, Proposition 23.1]), and hence $r(DP(0)) = r(Z(\omega)) = e^{\lambda_0 \omega}$. If $\lambda_0 \leqslant 0$, then $r(DP(0)) \leqslant 1$. By [25, Lemma 1] and the proof of [25, Theorem 2.2], P has no fixed point in $C^+(\bar{\Omega}, R^n)\backslash\{0\}$. Clearly, every steady state of (2.5) is a fixed point of the ω-time map P. Then (2.5) admits at most one positive steady state, and, if $\lambda_0 \leqslant 0$, (2.5) has no steady state in $C^+(\bar{\Omega}, R^n)\backslash\{0\}$.

Let μ_0 be the principal eigenvalue of the eigenvalue problem associated with (2.3) and let $v \gg 0$ in $C(\bar{\Omega}, R^n)$ be its corresponding eigenfunction. Let $\psi \gg 0$ in X_τ be as in Claim 2 of the proof of Theorem 2.1. In the case where $\lambda_0 \leqslant 0$, by [16, Theorem 3.1], then $\mu_0 \leqslant 0$. For any $\phi \in X_\tau^+$, by the subhomogeneity of $S(t)$,

$$S(t)\,\phi = \lim_{\alpha \to 0+} \frac{\alpha S(t)\,\phi}{\alpha} \leqslant \lim_{\alpha \to 0+} \frac{S(t)(\alpha\phi) - S(t)\,\hat{0}}{\alpha} = DS(t)(\hat{0})\,\phi, \qquad t \geqslant |\tau|.$$

For any $\beta > 0$, given $t_0 \geqslant (n+1)\,|\tau|$, by Claim 2 of the proof of Theorem 2.1, $r(DS(t_0)(\hat{0})) = e^{\mu_0 t_0} \leqslant 1$ and

$$S(t_0)(\beta\psi) \leqslant DS(t_0)(\hat{0})(\beta\psi) = \beta DS(t_0)(\hat{0})(\psi) = \beta e^{\mu_0 t_0}\psi \leqslant \beta\psi.$$

By the compactness and monotonicity of $S(t_0)$, $(S(t_0))^n\,(\beta\psi) = S(nt_0)(\beta\psi) \to \phi^*(n \to \infty)$ and $\phi^* = S(t_0)\,\phi^*$. It easily follows that $S(t)\,\phi^*$ is t_0-periodic with respect to $t \geqslant 0$ and $\lim_{t \to \infty} \|S(t)(\beta\psi)) - S(t)(\phi^*)\| = 0$. Then the omega-limit set of $\beta\psi$ for semiflow $S(t)$ is a t_0-periodic orbit $\gamma^+(\phi^*) = \{S(t)\,\phi^*;\, t \geqslant 0\}$, i.e., $\omega(\beta\psi) = \gamma^+(\phi^*)$. Since $t_0 \geqslant (n+1)\,|\tau|$ is arbitrary, $\omega(\beta\psi)$ has any $t_0 \geqslant (n+1)\,|\tau|$ as a period, which implies that $\omega(\beta\psi)$ is a steady state of $S(t)$. By the nonexistence of steady states in $X_\tau^+\backslash\{\hat{0}\}$,

$\omega(\beta\psi)=\hat{0}$, i.e., $\lim_{t\to\infty} S(t)(\beta\psi)=\hat{0}$. Therefore for any $\phi \geqslant \hat{0}$, there exists $\beta>0$ such that $\hat{0}\leqslant\phi\leqslant\beta\psi$ and hence $\hat{0}\leqslant S(t)\,\phi\leqslant S(t)(\beta\psi)$. Then $\lim_{t\to\infty} u_t(\phi)=\lim_{t\to\infty} S(t)\,\phi=\hat{0}$.

In the case where $\lambda_0>0$, by Theorem 2.1, either

(b_1) for any $\phi\in X_\tau^+\backslash\{\hat{0}\}$, $\lim_{t\to\infty}\|u_t(\phi)\|=+\infty$, or alternatively,

(b_2) (2.1) admits a unique positive steady state $\psi^*\gg 0$ in $C(\bar{\Omega}, R^n)$ such that for any $\hat{0}<\phi\leqslant\widehat{\psi^*}$, $\lim_{t\to\infty} u_t(\phi)=\widehat{\psi^*}$.

In the latter case, for any $\beta>1$, given $t_0\geqslant|\tau|$, by the subhomogeneity of $S(t)(t\geqslant 0)$,

$$\widehat{\psi^*}\leqslant S(t_0)(\beta\widehat{\psi^*})\leqslant\beta S(t_0)(\widehat{\psi^*})=\beta\widehat{\psi^*}.$$

By a similar argument to that in case (a), it follows that $S(t)(\beta\widehat{\psi^*})$ converges to a steady state $\widehat{\phi^*}$ and $\widehat{\phi^*}\geqslant\widehat{\psi^*}\gg 0$. Then by Claim 2 on the uniqueness of the steady state of $S(t)$ in $X_\tau^+\backslash\{\hat{0}\}$, $\lim_{t\to\infty} S(t)(\beta\widehat{\psi^*})=\widehat{\psi^*}$. Therefore for any $\phi\geqslant\widehat{\psi^*}$, there exists $\beta>1$ such that $\widehat{\psi^*}\leqslant\phi\leqslant\beta\widehat{\psi^*}$, and hence $\widehat{\psi^*}\leqslant S(t)\,\phi\leqslant S(t)(\beta\widehat{\psi^*})$. Then $\lim_{t\to\infty} S(t)\,\phi=\widehat{\psi^*}$. Since for any $\phi>\hat{0}$, there exist $\hat{0}<\phi_1\leqslant\widehat{\psi^*}$ and $\widehat{\psi^*}\leqslant\phi_2$ such that $\phi_1\leqslant\phi\leqslant\phi_2$ and hence $S(t)(\phi_1)\leqslant S(t)\,\phi\leqslant S(t)(\phi_2)$. It follows that $\lim_{t\to\infty} S(t)\,\phi=\widehat{\psi^*}$.

This completes the proof.

Consider the quasimonotone reaction-diffusion system

$$\begin{cases}\dfrac{\partial u(x,t)}{\partial t}=D\Delta u(x,t)+F(x,u(x,t)), & t>0, \quad x\in\Omega\\[2mm] \dfrac{\partial u(x,t)}{\partial \nu}+\alpha(x)\,u(x,t)=0, & t>0, \quad x\in\partial\Omega,\end{cases}\tag{2.9}$$

where $D=\operatorname{diag}(d_1, ..., d_n)$ with $d_i>0$, $1\leqslant i\leqslant n$, Δ is the Laplacian operator, and $F\colon\bar{\Omega}\times R^n\to R^n$ is twice continuously differentiable in (x,u).

By Theorems 2.1 and 2.2, we have the following results, respectively.

Corollary 2.1. *Assume that*

(1) *for each $x\in\bar{\Omega}$, $F(x,\cdot)$ is quasimonotone in $u\in R_+^n$, i.e., for any $u\in R_+^n$, $\partial F_i(x,u)/\partial u_j\geqslant 0$, $1\leqslant i\neq j\leqslant n$, and $F_u(x,0)=(\partial F_i(x,0)/\partial u_j)_{1\leqslant i,\,j\leqslant n}$ is irreducible*;

(2) *$F(x,0)\equiv 0$, and for each $x\in\bar{\Omega}$, $F_i(x,u)\geqslant 0$ for all $u\in R_+^n$ with some $u_i=0$, and for any $\phi\in C^+(\bar{\Omega}, R^n)$, the unique solution $u(t,\phi)$ of* (2.9) *with $u(0,\phi)=\phi$ exists globally on* $[0,+\infty)$;

(3) $\lambda_0=\lambda_0(F_u(\cdot,0))>0$.

Then either

(a) *for any* $\phi \in C^+(\bar{\Omega}, R^n)\backslash\{\hat{0}\}$, $\lim_{t\to\infty} \|u(t,\phi)\| = +\infty$; *or alternatively,*

(b) (2.9) *admits a positive steady state* $\psi^* \gg 0$ *in* $C(\bar{\Omega}, R^n)$ *such that for any* $0 < \phi \leqslant \psi^*$, $\lim_{t\to\infty} u(t,\phi) = \psi^*$ *in* $C(\bar{\Omega}, R^n)$. *Moreover, for any* $\phi > 0$, $\liminf_{t\to\infty} u(t,\phi) \geqslant \psi^*$.

COROLLARY 2.2. *Assume that*

(1) *for each* $x \in \bar{\Omega}$, $F(x, \cdot)$ *is quasimonotone in* $u \in R^n_+$, *and* $F_u(x,u) = (\partial F_i(x,u)/\partial u_j)_{1 \leqslant i,\, j \leqslant n}$ *is irreducible for any* $u \in R^n_+$;

(2) $F(x, 0) \equiv 0$, *and for each* $x \in \bar{\Omega}$, $F_i(x, u) \geqslant 0$ *for all* $u \in R^n_+$ *with some* $u_i = 0$;

(3) *for each* $x \in \bar{\Omega}$, $F(x, \cdot)$ *is strictly subhomogeneous on* R^n_+, *i.e., for any* $\alpha \in (0, 1)$ *and any* $u \gg 0$, $F(x, \alpha u) > \alpha F(x, u)$.

Then the following trichotomy holds:

(a) *if* $\lambda_0 = \lambda_0(F_u(\cdot, 0)) \leqslant 0$, *then* $u = 0$ *is globally attractive in* $C^+(\bar{\Omega}, R^n)$ *for* (2.9);

(b) *if* $\lambda_0 > 0$, *then either*

(b_1) *for any* $\phi \in C^+(\bar{\Omega}, R^n)\backslash\{\hat{0}\}$, $\lim_{t\to\infty} \|u(t, (\phi)\| = +\infty$; *or alternatively,*

(b_2) (2.9) *admits a unique positive steady state* $\psi^* \gg 0$ *and* $u = \psi^*$ *is globally attractive in* $C^+(\bar{\Omega}, R^n)\backslash\{0\}$ *for* (2.9).

Remark 2.1. Corollary 2.2 generalizes a threshold result [25, Theorem 3.4] for the scalar Kolmogorov reaction-diffusion equation and a related result [14, Theorem 7] on the global attractivity for the quasimonotone reaction-diffusion system subject to a Dirichlet boundary condition and with reaction term $F(\cdot)$ satisfying both the strict subhomogeneity and the one-sided monotone Lipschitz condition, i.e., for each $\rho > 0$, there is an $n \times n$ quasi-monotone and irreducible matrix C_ρ such that $F(\xi) - F(\eta) \geqslant C_\rho(\xi - \eta)$ for all $\xi \geqslant \eta \geqslant 0$ in R^n with $|\xi|, |\eta| < \rho$ (see (C3) and (C4) in [14, Theorem 7]).

3. SOME APPLICATIONS

From Section 2, we see that the principal eigenvalue of the weakly coupled linear elliptic eigenvalue problem plays a key role in the applications of our results. In this section, we first discuss some properties of the principal eigenvalue and then give two applications.

Consider the following eigenvalue problem

$$\begin{cases} D\Delta w(x) + M(x)\, w(x) = \lambda w(x), & x \in \Omega \\ \dfrac{\partial w(x)}{\partial \nu} + \alpha(x)\, w(x) = 0, & x \in \partial\Omega, \end{cases} \tag{3.1}$$

where $D = \operatorname{diag}(d_1, ..., d_n)$ with $d_i > 0$, $1 \leqslant i \leqslant n$, $\alpha(\cdot) = (\alpha_1(\cdot), ..., \alpha_n(\cdot))^T$: $\bar{\Omega} \to R^n_+$ is continuously differentiable, $M(x) = (m_{ij}(x))_{1 \leqslant i,\, j \leqslant n}$ is a continuous $n \times n$ matrix satisfying the cooperative and irreducible condition (CR). Then, by [21, Theorem 7.6.1], (3.1) admits a unique principal eigenvalue $\lambda_0 = \lambda_0(D, M(\cdot), \alpha(\cdot))$ with a corresponding eigenfunction $w_0(x) \gg 0$ for all $x \in \bar{\Omega}$.

We have the following result on the monotonicity of λ_0.

PROPOSITION 3.1. *Let $A(x) = (a_{ij}(x))$ and $B(x) = (b_{ij}(x))$ be two continuous $n \times n$ matrices satisfying (CR), and $\alpha(\cdot), \beta(\cdot) \in C^1(\bar{\Omega}, R^n_+)$. If $A(\cdot) \leqslant B(\cdot)$ (i.e., $a_{ij}(x) \leqslant b_{ij}(x)$, $x \in \bar{\Omega}$, $1 \leqslant i, j \leqslant n$) and $\alpha(\cdot) \geqslant \beta(\cdot)$ (i.e., $\alpha_i(x) \geqslant \beta_i(x)$, $x \in \bar{\Omega}$, $1 \leqslant i \leqslant n$), then $\lambda_0(D, A(\cdot), \alpha(\cdot)) \leqslant \lambda_0(D, B(\cdot), \beta(\cdot))$.*

Proof. Let $w_0(x) \gg 0$ be the principal eigenfunction corresponding to $\lambda_2 = \lambda_0(D, B(\cdot), \beta(\cdot))$. Then $w(x, t) \equiv e^{\lambda_2 t} w_0(x)$ satisfies the following linear parabolic system

$$\begin{cases} \dfrac{\partial w}{\partial t} = D\Delta w + B(x)\, w, & t > 0, \quad x \in \Omega \\ \dfrac{\partial w}{\partial \nu} + \beta(x)\, w = 0, & t > 0, \quad x \in \partial\Omega. \end{cases} \tag{3.2}$$

Since $A(\cdot) \leqslant B(\cdot)$ and $\alpha(\cdot) \geqslant \beta(\cdot)$, $w(x, t)$ satisfies the following differential inequality

$$\begin{cases} \dfrac{\partial w}{\partial t} \geqslant D\Delta w + A(x)\, w, & t > 0, \quad x \in \Omega \\ \dfrac{\partial w}{\partial \nu} + \alpha(x)\, w \geqslant 0, & t > 0, \quad x \in \partial\Omega. \end{cases} \tag{3.3}$$

Define $T(t)\colon C(\bar{\Omega}, R^n) \to C(\bar{\Omega}, R^n)$, $t \geqslant 0$, by $T(t)\,\phi = w(t, \phi)$, where $w(t, \phi)$ is the unique solution of the linear parabolic system

$$\begin{cases} \dfrac{\partial w}{\partial t} = D\Delta w + A(x)\, w, & t > 0, \quad x \in \Omega \\ \dfrac{\partial w}{\partial \nu} + \alpha(x)\, w = 0, & t > 0, \quad x \in \partial\Omega \end{cases}$$

with $w(0, \phi)=\phi$. Then, by [21, Theorems 7.3.1 and 7.4.1], $T(t)\colon C(\bar{\Omega}, R^n) \to C(\bar{\Omega}, R^n)$, $t \geqslant 0$, is a semiflow and for any $t>0$, $T(t)$ is compact and strongly positive, i.e., for any $\phi>0$, $T(t)\,\phi \gg 0$. Therefore, by a comparison theorem on weakly coupled parabolic systems (see, e.g., [14, Lemma 3], [3, Theorem B.19] and [21, Theorem 7.3.4])

$$w(t, \cdot)=e^{\lambda_2 t} w_0(\cdot) \geqslant T(t)\, w_0, \qquad t \geqslant 0,$$

that is,

$$(e^{\lambda_2 t} I - T(t))\, w_0 \geqslant 0, \qquad t \geqslant 0.$$

Since $w_0 \gg 0$ in $C(\bar{\Omega}, R^n)$, by the Krein–Rutman theorem (see, e.g., [1, Theorem 3.2] or [10, Theorem 7.3]),

$$e^{\lambda_2 t} \geqslant r(T(t))=e^{\lambda_1 t}, \qquad t>0,$$

where $r(T(t))$ is the spectral radius of $T(t)$ and $\lambda_1=\lambda_0(D, A(\cdot), \alpha(\cdot))$. Therefore $\lambda_2=\lambda_0(D, B(\cdot), \beta(\cdot)) \geqslant \lambda_1=\lambda_0(D, A(\cdot), \alpha(\cdot))$.

This completes the proof.

If we assume that $M(\cdot)$ is a constant $n \times n$ cooperative and irreducible matrix and that the same boundary condition is imposed on all components, we further have the following result.

PROPOSITION 3.2. *Assume that* $B=(b_{ij})_{1 \leqslant i, j \leqslant n}$ *is cooperative (i.e.,* $b_{ij} \geqslant 0$, $i \neq j$*) and irreducible, and that* $\beta(x)=(\alpha(x), ..., \alpha(x))^T$ *with* $\alpha(\cdot) \in C^1(\bar{\Omega}, R_+)$. *Then*

$$\lambda_0(D, B, (\cdot))=\max\{\operatorname{Re} \lambda;\ \det[\lambda I+\lambda_\alpha D-B]=0\},$$

where λ_α *is the smallest eigenvalue of the scalar eigenvalue problem*

$$\begin{cases} D\, \Delta w(x)+\lambda w(x)=0, & x \in \Omega, \\ \dfrac{\partial w(x)}{\partial \nu}+\alpha(x)\, w(x)=0, & x \in \partial\Omega. \end{cases} \tag{3.4}$$

Proof. By [1], there is an eigenfunction $w_\alpha(x) \geqslant 0 (x \in \bar{\Omega})$ corresponding to λ_α. Since the matrix $B-\lambda_\alpha D$ is a cooperative and irreducible one, by the Krein–Rutman theorem (or the Perron–Frobenius theorem in our present finite dimensional case), $B-\lambda_\alpha D$ admits a unique principal eigenvalue

$$\bar{\lambda}_0=\max\{\operatorname{Re} \lambda;\ \det[\lambda I+\lambda_\alpha D-B]=0\}$$

with a corresponding eigenvector $\xi \gg 0$ in R^n, i.e., $(B-\lambda_\alpha D)\,\xi=\bar{\lambda}_0\xi$. Then $w_\xi(t)=e^{\bar{\lambda}_0 t}\xi$ satisfies the n-dimensional system of ordinary differential equations

$$\frac{dw}{dt}=(B-\lambda_\alpha D)\,w, \qquad t\geqslant 0. \tag{3.5}$$

Let $v(x,t)=w_\alpha(x)\cdot w_\xi(t)$, $t\geqslant 0$, $x\in\bar{\Omega}$. Then it easily follows that $v(x,t)$ satisfies the linear parabolic system

$$\begin{cases} \dfrac{\partial v}{\partial t}=D\,\Delta v+Bv, & t>0, \quad x\in\Omega \\ \dfrac{\partial v}{\partial \nu}+\beta(x)v=0, & t>0, \quad x\in\partial\Omega. \end{cases} \tag{3.6}$$

Let $T(t)\colon C(\bar{\Omega},R^n)\to C(\bar{\Omega},R^n)$, $t\geqslant 0$, be the solution semiflow generated by (3.6). Then $T(t)(v(\cdot,0))=v(\cdot,t)$, $t\geqslant 0$. Therefore

$$T(t)(w_\alpha(\cdot)\xi)=w_\alpha(\cdot)\,w_\xi(t)=w_\alpha(\cdot)(e^{\bar{\lambda}_0 t}\xi)=e^{\bar{\lambda}_0 t}(w_\alpha(\cdot)\,\xi),\ t\geqslant 0.$$

Since $w_\alpha(\cdot)\xi\gg 0$ in $C(\bar{\Omega},R^n)$ and $T(t)$ is compact and strongly positive for each $t>0$ (see [21, Theorems 7.3.1 and 7.4.1]), the Krein–Rutman theorem ([1, Theorem 3.2] or [10, Theorem 7.3]) implies that $e^{\bar{\lambda}_0 t}=r(T(t))=e^{\lambda_0 t}$, $t>0$, where $\lambda_0=\lambda_0(D,B,\beta(\cdot))$. Therefore $\lambda_0=\bar{\lambda}_0$.

This completes the proof.

Remark 3.1. If the boundary condition $\partial w/\partial\nu+\alpha(x)\,w=0$ in (3.1) and (3.4) is replaced by the Dirichlet boundary condition $w=0$, the conclusion of Proposition 3.2 is also valid. In this case, system (3.6) generates a semiflow $T(t)$, $t\geqslant 0$, on $C_0^1(\bar{\Omega},R^n)$ and $T(t)\colon C_0^1(\bar{\Omega},R^n)\to C_0^1(\bar{\Omega},R^n)$ is compact and strongly positive for each $t>0$ (see [21, Theorems 7.3.1 and 7.4.1 and Section 7.4]. Therefore an argument similar to that in Proposition 3.2 applies.

We now consider a class of delayed reaction-diffusion equations models of single species growth (see [7] for the model without diffusion),

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t}=d\,\Delta u(x,t)+u(x,t)\;g(x,u(x,t),u(x,t-\tau)), & t>0, \quad x\in\Omega \\ \dfrac{\partial u(x,t)}{\partial \nu}+\alpha(x)\,u(x,t)=0, & t>0, \quad x\in\partial\Omega, \end{cases} \tag{3.7}$$

where $g\colon\bar{\Omega}\times R^2\to R$ is a continuously differentiable function, $d>0$, $\tau>0$ and $\alpha(\cdot)\in C^1(\bar{\Omega},R_+)$. We assume that the single species diffuses in the habitat Ω, that $u(x,t)$ represents the population density at the point x and time t, and that the per capita growth is a density-dependent function of the current population size as well as that of τ units of time earlier.

Let $\lambda_0 = \lambda_0(g(\cdot, 0, 0))$ be the principal eigenvalue of the scalar eigenvalue problem

$$\begin{cases} d\,\Delta w(x) + g(x, 0, 0)\, w(x) = \lambda w(x), & x \in \Omega \\ \dfrac{\partial w(x)}{\partial v} + \alpha(x)\, w(x) = 0, & x \in \partial\Omega. \end{cases} \tag{3.8}$$

Then we have the following threshold result on the global dynamics of (3.7).

PROPOSITION 3.3. *For any $\phi \in X^+ = C(\bar{\Omega} \times [-\tau, 0], R^+)$, let $u(x, t, \phi)$ be the unique solution of* (3.7) *satisfying $u(x, \theta, \phi) = \phi(x, \theta)$, $x \in \bar{\Omega}$, $\theta \in [-\tau, 0]$. Assume that*

(i) *$\partial g(x, u, v)/\partial v \geqslant 0$ for all $x \in \bar{\Omega}$, $(u, v) \in R^2_+$.*

(ii) *$\partial g(x, u, u)/\bar{\Omega} u < 0$ for all $x \in \bar{\Omega}$ and $u \geqslant 0$, and there exits $K_0 > 0$ such that $g(x, K_0, K_0) \leqslant 0$ for all $x \in \bar{\Omega}$.*

(a) *If $\lambda_0(g(\cdot, 0, 0)) \leqslant 0$, then for every $\phi \in X^+$, $\lim_{t \to \infty} u(x, t, \phi) = 0$ uniformly for $x \in \bar{\Omega}$.*

(b) *If $\lambda_0(g(\cdot, 0, 0)) > 0$, then* (3.7) *has a unique positive equilibrium $u^*(x)$ such that for every $\phi \in X^+$ with $\phi(\cdot, 0) \not\equiv 0$, $\lim_{t \to \infty} u(x, t, \phi) = u^*(x)$ uniformly for $x \in \bar{\Omega}$.*

Proof. Let $F(x, \phi) = \phi(0)\, g(x, \phi(0), \phi(-\tau))$, $\phi \in C([-\tau, 0]\, R)$. By assumption (i), F satisfies (QM) on $C^+([-\tau, 0], R)$. Since $L(x)\,\psi = d_\phi F(x, \hat{0})\,\psi = g(x, 0, 0)\,\psi(0)$, $\psi \in C([-\tau, 0]\, R)$, does not satisfy (R) and there is no subhomogeneous assumption on $F(x, \cdot)$, we cannot use Theorems 2.1 and 2.2 directly. For any $K \geqslant K_0$, by assumptions (i) and (ii) and [16, Proposition 1.3], the interval $I = [0, K]$ is positively invariant for (3.7). Then for any $\phi \in X^+$, $u(t, \phi) > 0$ exists globally on $[0, \infty)$ and $u(t, \phi)$ is bounded. Moreover, if $\phi \in X^+$ with $\phi(\cdot, 0) \not\equiv 0$, then the classical maximum principle arguments give $u(t, \phi) \gg 0$ in $C(\bar{\Omega})$ for all $t > 0$. If $\phi \in X^+$ with $\phi(\cdot, 0) \equiv 0$, then $u(t, \phi) \equiv 0$ for all $t > 0$. For any $\phi \in X^+$, by assumption (i), $u(x, t, \phi)$ satisfies

$$\begin{cases} \dfrac{\partial u(x, t)}{\partial t} \geqslant d\,\Delta u(x, t) + u(x, t)\; g(x, u(x, t), 0), & t > 0, \quad x \in \Omega \\ \dfrac{\partial u(x, t)}{\partial v} + \alpha(x)\, u(x, t) = 0, & t > 0, \quad x \in \partial\Omega. \end{cases}$$

By the boundedness of $u(t, \phi)$, the standard comparison theorem argument and Corollary 2.1, it then follows that if $\lambda_0(g(\cdot, 0, 0)) > 0$, Eq. (3.7) is uniformly persistent, i.e., there exists $\delta > 0$ such that for any $\phi \in X^+$ with $\phi(\cdot, 0) \not\equiv 0$, there exists a $T = T(\phi) > 0$ such that $u(x, t, \phi) \geqslant \delta$ for all $t \geqslant T$ and $x \in \bar{\Omega}$. By assumption (ii), $g(x, u, u)$ is strictly decreasing in u, and hence, for each $x \in \bar{\Omega}$, $f(x, u) \equiv ug(x, u, u)$ is strictly subhomogeneous on R_+. Then, by Corollary 2.2, the reaction-diffusion equation

$$\begin{cases} \dfrac{\partial u(x, t)}{\partial t} = d\, \Delta u(x, t) + u(x, t)\, g(x, u(x, t), u(x, t)), & t > 0, \quad x \in \Omega \\ \dfrac{\partial u(x, t)}{\partial \nu} + \alpha(x)\, u(x, t) = 0, & t > 0, \quad x \in \partial\Omega \end{cases}$$

and hence (3.7) admits no positive steady state in the case where $\lambda_0(g(\cdot, 0, 0)) \leqslant 0$, and has a unique positive steady state $u^*(x)$ in the case where $\lambda_0(g(\cdot, 0, 0)) > 0$. Define $S(t)\, \phi = u_t(\phi)$, $t \geqslant 0$, $\phi \in X^+$. Then, by [16, Proposition 1.4], $S(t)\colon X^+ \to X^+$, $t \geqslant 0$, is a monotone semiflow. For any given $K \geqslant \max(K_0, \|u^*(\cdot)\|)$, let

$$Y = \{\phi \in X^+;\, 0 \leqslant \phi(x, \theta) \leqslant K,\, x \in \bar{\Omega},\, \theta \in [-\tau, 0]\},$$

$$\partial Y_0 = \{\phi \in Y;\, \phi(\cdot, 0) \equiv 0\}, \qquad \text{and} \qquad Y_0 = \{\phi \in Y;\, \phi(\cdot, 0) \not\equiv 0\}.$$

Then $Y = Y_0 \cup \partial Y_0$, ∂Y_0 is a closed set in Y and $S(t)\colon Y \to Y$ satisfies

$$S(t)\colon Y_0 \to Y_0, \qquad S(t)\colon \partial Y_0 \to \partial Y_0.$$

Clearly, $S(t)\colon Y \to Y$ is dissipative and compact for $t > \tau$. By [8, Theorem 3.4.8], there exists a global attractor A. In the case where $\lambda_0(g(\cdot, 0, 0)) \leqslant 0$, A contains only one equilibrium $\hat{0}$, and hence, by [11, Theorem 3.3], $\hat{0}$ attracts any point in Y, i.e., for any $\phi \in Y$, $\lim_{t \to \infty} u(x, t, \phi) = 0$ uniformly for $x \in \bar{\Omega}$. In the case where $\lambda_0(g(\cdot, 0, 0)) > 0$, $S(t)\colon Y \to Y$ is uniformly persistent with respect to $(Y_0, \partial Y_0)$, and hence, by [9, Theorem 3.2], there exists a global attractor A_0 in Y_0. Since A_0 contains only one equilibrium $u^*(\cdot)$, again by [11, Theorem 3.3] $u^*(\cdot)$ attracts any point in Y_0, i.e., for any $\phi \in Y_0$, $\lim_{t \to \infty} u(x, t, \phi) = u^*(x)$ uniformly for $x \in \bar{\Omega}$. Since K can be chosen to be large arbitrarily, the threshold result follows.

This completes the proof.

Remark 3.2. If we assume (i) and, instead of (ii), the following condition

(ii)′ there exists $K_0 > 0$ such that $g(x, u, u) \leqslant 0$ for all $x \in \bar{\Omega}$, $u \geqslant K_0$,

by the proof of Proposition 3.3 and Corollary 2.1, it then follows that Eq. (3.7) is uniformly persistent and admits at least one positive steady state provided $\lambda_0(g(\cdot, 0, 0)) > 0$.

Remark 3.3 As suggested in [7], an example of (3.7) is of the form

$$\begin{cases} \dfrac{\partial u}{\partial t} = d\,\Delta u + u(x,t)[a(x) - b(x)\,u(x,t) + c(x)\,u(x,t-\tau)], & t > 0, \quad x \in \Omega \\ \dfrac{\partial u(x,t)}{\partial \nu} + \alpha(x)\,u(x,t) = 0, & t > 0, \quad x \in \partial\Omega, \end{cases}$$

where $a(\cdot)$, $b(\cdot)$ and $c(\cdot)$ are positive functions with $b(x) > c(x)$, $x \in \bar{\Omega}$. This model implies that when the population size is small, growth is proportional to the size and when the population size is not so small, the positive feedback is $a(x) + c(x)\,u(x, t-\tau)$ while the negative feedback is $b(x)\,u(x,t)$. Circumstances of this type can arise when the resources are plentiful and the reproduction is by individuals of at least age τ units of time.

We then consider a delayed reaction-diffusion system modelling the spread of bacterial infections (see [4], [2] and [3, Section 5.2] for the model without delay),

$$\begin{cases} \dfrac{\partial u_1(x,t)}{\partial t} = d_1\,\Delta u_1(x,t) - a_{11} u_1(x,t) + a_{12} u_2(x,t), & t > 0, \quad x \in \Omega \\ \dfrac{\partial u_2(x,t)}{\partial t} = d_2\,\Delta u_2(x,t) + g(u_1(x,t-\tau)) - a_{22} u_2(x,t), & t > 0, \quad x \in \Omega \\ \dfrac{\partial u_i(x,t)}{\partial \nu} + \alpha_i(x)\,u_i(x,t) = 0, \qquad 1 \leqslant i \leqslant 2, & t > 0, \quad x \in \partial\Omega, \end{cases} \tag{3.9}$$

where $\tau \geqslant 0$, d_1, d_2, a_{11}, a_{12} and a_{22} are all positive constants, $g\colon R_+ \to R_+$ is a continuously differentiable function, $u_1(x,t)$ and $u_2(x,t)$ represent the population densities at the point x and time t of a bacterial population and of a human population infected by the bacteria respectively. Both bacteria and humans are assumed to diffuse; term $-a_{11}u_1$ arises because the bacterial population tends to die out in the absence of other factors; the term $a_{12}u_2$ represents the growth of the bacteria due to infected humans; the term $-a_{22}u_2$ arises because a certain population of the infected humans recover per unit time; $g(u_1(x, t-\tau))$ represents the growth of the infected humans at time t due to the bacterial population density at time $(t-\tau)$.

We have the following threshold result on the global dynamics of (3.9).

PROPOSITION 3.4. *Assume that*

(i) $g(0) = 0$ *and* $g'(z) = dg(z)/dz > 0$ *for* $z \geqslant 0$;

(ii) *$g(z)$ is strictly subhomogeneous on R_+, i.e., for any $z>0$ and any $\alpha \in (0, 1)$, $g(\alpha z) > \alpha g(z)$;*

(iii) *there exists $z_0 > 0$ such that $g(z_0)/z_0 \leqslant a_{11}a_{22}/a_{12}$.*

Let λ_0 be the principal eigenvalue of the eigenvalue problem (3.1) with $D = \mathrm{diag}(d_1, d_2)$, $\alpha(\cdot) = (\alpha_1(\cdot), \alpha_2(\cdot))^T$ and $M(x) = B$, where

$$B = \begin{pmatrix} -a_{11} & a_{12} \\ g'(0) & -a_{22} \end{pmatrix}.$$

For any $\phi = (\phi_1, \phi_2)^T \in Y^+ = C(\bar{\Omega} \times [-\tau, 0], R_+) \times C(\bar{\Omega}, R_+)$, let $u(x, t, \phi) = (u_1(x, t, \phi), u_2(x, t, \phi))$ be the unique solution of (3.9) satisfying $u_1(x, \theta, \phi) = \phi_1(x, \theta)$, $x \in \bar{\Omega}$, $\theta \in [-\tau, 0]$ and $u_2(x, 0, \phi) = \phi_2(x)$, $x \in \bar{\Omega}$.

(a) *If $\lambda_0 \leqslant 0$, then for every $\phi \in Y^+$, $\lim_{t\to\infty} u(x, t, \phi) = 0$ uniformly for $x \in \bar{\Omega}$.*

(b) *If $\lambda_0 > 0$, then (3.9) has a unique positive equilibrium $u^*(x) = (u_1^*(x), u_2^*(x))$ such that for every $\phi \in Y^+ \backslash \{0\}$, $\lim_{t\to\infty} u(x, t, \phi) = u^*(x)$ uniformly for $x \in \bar{\Omega}$.*

Proof. For any $\phi = (\phi_1, \phi_2) \in C([-\tau, 0], R_+) \times R_+$, let

$$F(\phi) = \begin{pmatrix} -a_{11}\phi_1(0) + a_{12}\phi_2 \\ g(\phi_1(-\tau)) - a_{22}\phi_2 \end{pmatrix}.$$

By the strict subhomogeneity of $g(\cdot): R_+ \to R$, it is easy to see that $F(\cdot): C([-\tau, 0], R_+) \times R_+ \to R^2$ is strictly subhomogeneous. It then follows that under assumptions (i) and (ii), $F(\phi)$ satisfies all the conditions in Theorem 2.2. Therefore it suffices to prove that system (3.9) has a bounded positive solution in the case where $\lambda_0 > 0$.

Indeed, let $\mu(\cdot): R_+ \to R$ be defined by

$$\mu(s) = \max\{\mathrm{Re}\,\lambda;\ \det[\lambda I + sD - B] = 0\}.$$

A direct computation shows that

$$\mu(x) = \tfrac{1}{2}[-(sd_1 + a_{11} + sd_2 + a_{22}) + \sqrt{(sd_1 + a_{11} - sd_2 - a_{22})^2 + 4a_{12}g'(0)}]. \tag{3.10}$$

Let $\Theta(\cdot): R_+ \to R$ be defined by

$$\Theta(s) = \frac{a_{12}g'(0)}{(a_{11} + d_1 s)(a_{22} + d_2 s)}, \qquad s \geqslant 0. \tag{3.11}$$

By (3.10) and (3.11), it easily follows that for each $s \in [0, \infty, \mu(s)$ and $(\Theta(s) - 1)$ have the same signs, i.e.,

$$\operatorname{sgn}(\mu(s)) = \operatorname{sgn}(\Theta(s) - 1), \qquad s \in [0, \infty), \tag{3.12}$$

where $\operatorname{sgn}(\cdot)\colon R \to R$ is the sign function defined by

$$\operatorname{sgn}(s) = \begin{cases} 1, & \text{if } s > 0 \\ 0, & \text{if } s = 0 \\ -1, & \text{if } s < 0. \end{cases}$$

Let $\alpha_m = \min_{x \in \partial\Omega} \min\{\alpha_1(x), \alpha_2(x)\}$, and let λ_m be the smallest eigenvalue of the scalar eigenvalue problem (3.4) with $\alpha(x)$ replaced by α_m. Then $\lambda_m \geqslant 0$ and, by the definition of $\mu(\cdot)$ and Propositions 3.1 and 3.2,

$$\lambda_0 \leqslant \lambda_0(D, B, \alpha_m) = \mu(\lambda_m). \tag{3.13}$$

In the case where $\lambda_0 > 0$, we have $\mu(\lambda_m) > 0$. Then, by (3.12), $\Theta(\lambda_m) > 1$, and hence, $a_{12} g'(0)/a_{11} a_{22} > 1$. Therefore, by assumption (iii),

$$\lim_{z \to 0+} \frac{g(z)}{z} = g'(0) > \frac{a_{11} a_{22}}{a_{12}} \geqslant \frac{g(z_0)}{z_0}.$$

Then there exists $u_1^* > 0$ such that $g(u_1^*)/u_1^* = a_{11} a_{22}/a_{12}$. Let $u_2^* = a_{11} u_1^*/a_{12}$. Then $u_2^* > 0$ and $u^* = (u_1^*, u_2^*)$ satisfies $F(u^*) = 0$. Clearly, $u(t) = (u_1^*, u_2^*)$ is a supersolution of (3.9), and hence, the comparison theorem implies that $0 < u(x, t, u^*) \leqslant u^*$, $t \geqslant 0$, $x \in \bar{\Omega}$. Therefore (3.9) has a bounded positive solution.

This completes the proof.

Remark 3.4. Let $\alpha_M = \max_{x \in \partial\Omega} \max\{\alpha_1(x), \alpha_2(x)\}$, and let λ_M be the smallest eigenvalue of the scalar eigenvalue problem (3.4) with $\alpha(x)$ replaced by α_M. Then by Propositions 3.1 and 3.2 and (3.13),

$$\mu(\lambda_M) = \lambda_0(D, B, \alpha_M) \leqslant \lambda_0 = \lambda_0(D, B, \alpha(\cdot)) \leqslant \lambda_0(D, B, \alpha_m) = \mu(\lambda_m). \tag{3.14}$$

In particular, if the Neumann boundary condition is imposed on both u_1 and u_2, i.e., $\alpha_1(\cdot) \equiv \alpha_2(\cdot) \equiv 0$, then $\lambda_M = \lambda_m = 0$, and hence $\lambda_0 = \mu(0)$.

Remark 3.5. For the reaction-diffusion system (3.9) without delay (i.e., $\tau = 0$), [3] and [4] introduced two threshold parameters $\Theta_M = \Theta(\lambda_M)$ and $\Theta_m = \Theta(\lambda_m)$ and proved that conclusion (a) holds for $\Theta_m < 1$ (see [4, Theorem 4.2]) and conclusion (b) holds for $\Theta_M > 1$ (see [4, Theorem 5.6]) with the assumptions (ii) and (iii) replaced by (ii)′ $g(\cdot): R_+ \to R_+$ is strictly subhomogeneous; and for any $\rho > 0$, there exists $k_\rho > 0$ such that for any $0 \leqslant z_1 \leqslant z_2 < \rho$, $g(z_2) - g(z_1) \geqslant k_\rho(z_2 - z_1)$; and

(iii)′ $\limsup_{z \to \infty} g(z)/z < a_{11} a_{22}/a_{12}$ respectively. A simple sufficient condition suggested in [3, Theorems 5.1 and 5.5] for (ii)″ to hold is that

(ii)″ $g(\cdot): R_+ \to R_+$ is twice continuously differentiable and $g''(z) = d^2 g(z)/dz^2 < 0$ for all $z > 0$.

Clearly, both (ii)′ and (iii)′ are stronger than (ii) and (iii), respectively. Moreover, by (3.12) and (3.14), $\Theta_m < 1$ implies $\lambda_0 < 0$, and $\Theta_M > 1$ implies $\lambda_0 > 0$. Therefore our Proposition 3.4 implies the main results in [4, Theorem 4.2 and 5.6].

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